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Chapter 1

Complex Numbers and its Applications

1.1 Complex Plane

1.1.1 Introduction

Considering the equation $x^2 + 1 = 0$ and finding the inability of mathematicians to solve equation of such type where square root of a negative real number comes to play because till then no one has any idea about the squareroot of negative real numbers. Owing to this difficulty the idea of the imaginary numbers is developed and mathematicians created a imaginary number called i(iota) for the square root of -1 i.e., $i = \sqrt{-1}$. With the help of this number now the square root of every real number is known to us which ultimately led to the new set of numbers known as complex numbers.

1.1.2 i and its powers

i, called iota, is a number whose square is -1. Since we know that there is no such real number whose square is -1, so $i \notin \mathbb{R}$ and $i^2 = -1$. Also, $(-i)(-i) = (-i)^2 = (i)^2 = -1$, so $-i$ is also a square root of -1.

Powers of i

$i^2 = i.i = \sqrt{-1} \times \sqrt{-1} = -1$

$i^3 = i^2.i = -1.i = -i$

$i^4 = i^2.i^2 = (-1).(-1) = 1$

$i^5 = i^4.i = 1.i = i$ and so on.

$\therefore$ The only possible values of $i^n, n \in \mathbb{N}$ is $1, -1, i, -i$

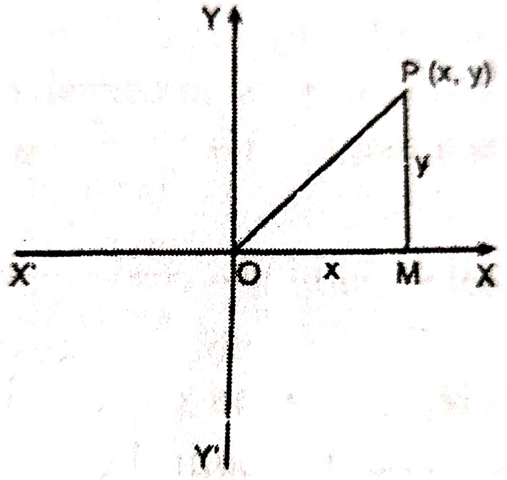
1.1.3 Complex Number

Definition A number of the form $a+ib$, where a and b are real numbers and $i = \sqrt{-1}$, is called a complex number where a is called a real part and b is called imaginary part of the complex number.

The number z is generally used to denote a complex number i.e., $z = a + ib$. Moreover the real part is denoted as $Re(z) = a$ and imaginary part is denoted as $Im(z) = b$. If $Re(z) = 0$, then the complex number is called the purely imaginary number and if $Im(z) = 0$, then the complex number is called purely real. The set of complex numbers is denoted by $\mathbb{C}$.

Geometrical Interpretation of a Complex Number

The complex number $z = x + iy$ is geometrically represented by a point P(x,y) (say) in a XY-plane. This XY-plane is called Complex plane or Argand plane or Gaussian plane.The X-axis of the complex plane is called the Real axis and the Y-axis of the complex plane is called the Imaginary axis. Thus, every complex number is represented by unique point in the complex plane and conversely to every point in the plane there is a unique complex number associated with it. Thus there is one-one correspondence between set of points in a plane and the set of complex numbers.



Equality of Complex Numbers: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers then $z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$.

1.1.4 Algebra of Complex numbers

1. Addition of two Complex numbers: If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers, then their addition(or sum) is defined by

$$z_1 + z_2 = (a_1 + ib_1) + (a_1 + ib_1) = (a_1 + a_2) + i(b_1 + b_2)$$

2. Difference of two Complex numbers: If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers, then their difference is defined by

$$z_1 - z_2 = (a_1 + ib_1) - (a_1 + ib_1) = (a_1 - a_2) + i(b_1 - b_2)$$

3. Multiplication (Product) of two Complex numbers: If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers, then their product is defined by

$$z_1 + z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

4. Division in Complex numbers: A complex number $a + ib$ is said to be divisible by a non-zero complex number $c + id$ if $\exists$ a complex number $x + iy$ s.t. $a + ib = (x + iy)(c + id)$
Now $(x + iy)(c + id) = a + ib$
$\implies (cx - dy) + i(dx + cy) = a + ib$
$\implies cx - dy = a$
$\quad dx + cy = b$
$\implies cx - dy - a = 0$
$\quad dx + cy - b = 0$
$\implies \frac{x}{bd+ac} = \frac{y}{-ad+bc} = \frac{1}{c^2+d^2}$
$\implies x = \frac{ac+bd}{c^2+d^2}$ and $y = \frac{-ad+bc}{c^2+d^2}$
Thus, $\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{-ad+bc}{c^2+d^2}$

Theorem 1.1.1. *Prove that the set of complex numbers $\mathbb{C}$ is a field under usual addition and multiplication.*

Proof. To prove that $\mathbb{C}$ is a field, we have to first prove that it is an abelian group under addition. Secondly, it will satisfy the 5 properties under multiplication and lastly the distributive property.

A. Properties under Addition

1. Closure Property: Since the sum of two complex numbers is a complex number.

$\therefore \mathbb{C}$ is closed under addition.

2. Commutative Property: Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then

$$\begin{aligned} z_1 + z_2 &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \\ &= (c + a) + i(d + b) \qquad [\because \text{Addition of real numbers is commutative}] \\ &= z_2 + z_1 \end{aligned}$$

$\therefore z_1 + z_2 = z_2 + z_1 \quad \forall z_1, z_2 \in \mathbb{C}$.

Hence addition of complex numbers is commutative.

3. Associative Property: Let $z_1 = a+ib, z_2 = c+id, z_3 = e+if \in \mathbb{C}$.

$$\begin{aligned} \therefore (z_1 + z_2) + z_3 &= [(a + ib) + (c + id)] + (e + if) \\ &= [(a + c) + i(b + d)] + (e + if) \\ &= [(a + c) + e] + i[(b + d) + f] \\ &= [a + (c + e)] + i[b + (d + f)] \; [\because \text{Addition of real numbers is associ} \\ &= (a + ib) + [(c + e) + i(d + f)] \\ &= (a + ib) + [(c + id) + (e + if)] \\ &= z_1 + (z_2 + z_3) \end{aligned}$$

$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \forall z_1, z_2, z_3 \in \mathbb{C}$

4. Existence of Additive Identity: The complex number $0 = 0 + i0$ is additive identity in $\mathbb{C}$.

Since for any complex number $z = a + ib$, we have

$$\begin{aligned} z + 0 &= (a + ib) + (0 + i0) \\ &= (a + 0) + i(b + 0) \\ &= a + ib = z \end{aligned}$$

Similarly $0 + z = z$.
Thus, $z + 0 = z = 0 + z \implies$ complex number $0 = 0 + i0$ is additive identity in $\mathbb{C}$.
5. Existence of Additive Inverse: For each complex number $a + ib$, $\exists$ complex number $-a - ib$ s.t.

$$\begin{aligned} (a + ib) + (-a - ib) &= (a - a) + i(b - b) \\ &= 0 + i0 \\ &= 0 \end{aligned}$$

$\therefore -a - ib$ is additive inverse of $a + ib$.
B. Properties under Multiplication
1. Closure Property: Since multiplication of two complex numbers is a complex number.
$\therefore \mathbb{C}$ is closed under multiplication.
2. Commutative Property: Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then
$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$

and

$$\begin{aligned} z_2 z_1 &= (c+id)(a+ib) = (ca-db)+i(cb+da) \\ &= (ac-bd)+i(bc+ad) \qquad [\because \text{multiplication of real numbers is commutative}] \\ &= (ac-bd)+i(ad+bc) \qquad [\because \text{addition of real numbers is commutative}] \end{aligned}$$

$\therefore z_1 z_2 = z_2 z_1 \quad \forall z_1, z_2 \in \mathbb{C}$.
Hence multiplication of complex numbers is commutative.

3. Associative Property: Let $z_1 = a+ib, z_2 = c+id, z_3 = e+if \in \mathbb{C}$.
Now

$$\begin{aligned} (z_1 z_2) z_3 &= [(a+ib)(c+id)](e+if) \\ &= [(ac-bd)+i(ad+bc)](e+if) \\ &= [(ac-bd)e-(ad+bc)f]+i[(ac-bd)f+(ad+bc)e] \\ &= (ace-bde-adf-bcf)+i(acf-bdf+ade+bce) \end{aligned}$$

and

$$\begin{aligned} z_1(z_2 z_3) &= (a+ib)[(c+id)(e+if)] \\ &= (a+ib)[(ce-df)+i(cf+de)] \\ &= [a(ce-df)-b(cf+de)]+i[a(cf+de)+b(ce-df)] \\ &= (ace-adf-bcf-bde)+i(acf+ade+bce-bdf) \end{aligned}$$

$\therefore (z_1 z_2) z_3 = z_1(z_2 z_3) \quad \forall z_1, z_2, z_3 \in \mathbb{C}$.
Hence multiplication of complex numbers is associative.

4. Existence of Multiplicative Identity: The complex number $1 = 1 + i0$ is multiplicative identity in $\mathbb{C}$.
Since for any $z = a + ib \in \mathbb{C}$, we have

$z.1 = (a + ib).(1 + i0) = (a.1 - 0.b) + i(a.0 + b.1) = a + ib = z.$
Similarly $1.z = 1$
$\therefore z.1 = z = 1.z \implies 1$ is multiplicative identity.

5. Existence of Multiplicative Inverse of non-zero complex numbers: Let $a + ib$ be any non-zero complex number, then a complex number $x = iy$ is called multiplicative inverse of $a+ib$ if $(a+ib)(x+iy) = 1$.

Now $(a + ib)(x + iy) = 1$
$\implies (ax - by) + i(bx + ay) = 1 + i0$
Equating real and imaginary parts, we get
$ax - by = 1$ and $bx + ay = 0$
i.e., $ax - by - 1 = 0$ and $bx + ay + 0 = 0$
$\implies \frac{x}{0+a} = \frac{y}{-b-0} = \frac{1}{a^2+b^2} \implies x = \frac{a}{a^2+b^2}, y = \frac{-b}{a^2+b^2}$
which exists finitely $\left[\because a + ib \neq 0 \implies \text{ a and b are not both zero } \implies a^2 + b^2 \neq 0\right]$
$\therefore$ multiplicative inverse of $a + ib$ is $\frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$
Thus every non-zero complex number possesses its multiplicative inverse.

C. Distributive Property

Let $z_1 = a + ib, z_2 = c + id, z_3 = e + if \in \mathbb{C}$.
Now

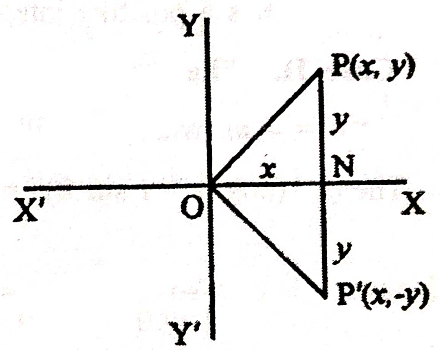
$$\begin{aligned}
z_1(z_2 + z_3) &= (a + ib)[(c + id) + (e + if)] \\
&= (a + ib)[(c + e) + i(d + f)] \\
&= [a(c + e) - b(d + f)] + i[b(c + e) + a(d + f)] \\
&= (ac + ae - bd - bf) + i(bc + be + ad + af)
\end{aligned}$$

and

$$\begin{aligned}z_1z_2 + z_1z_3 &= (a+ib)(c+id) + (a+ib)(e+if) \\ &= [(ac-bd)+i(ad+bc)] + [(ae-bf)+i(af+be)] \\ &= (ac-bd+ae-bf) + i(ad+bc+af+be)\end{aligned}$$

$\therefore z_1(z_2+z_3) = z_1z_2 + z_1z_3$
Hence distributive property hold in $\mathbb{C}$.
Hence $\mathbb{C}$ is a field. ■

Conjugate of a Complex Number
If $z = x+iy$ is a complex number, then the conjugate of a complex number z, denoted as $\overline{z}$, is defined as $\overline{z} = x - iy$. **Geometrically**, the conjugate of a complex number is the point in a complex plane which is the image of the complex number in the x-axis i.e., if $P = (x, y)$ be a point in the complex plane representing the complex number $z = x+iy$, then $P' = (x, -y)$ is the image of the point $P = (x, y)$ in the x-axis $\therefore P'$ represent the conjugate of the complex number z i.e., $x + i(-y) = x - iy = \overline{z}$.



Modulus of a Complex Number
If $z = x+iy$ is a complex number, then the modulus of the complex number z, denoted by $|z|$, is defined as $|z| = \sqrt{x^2+y^2}$

Properties of Conjugate and Modulus of a Complex Number

If z_1 and z_2 is a complex number, then

(i) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(ii) $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$

(iii) $z\overline{z} = |z|^2$

(iv) $z + \overline{z} = 2Re(z)$

(v) $z - \overline{z} = 2iIm(z)$

(vi) $\overline{(\frac{z_1}{z_2})} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

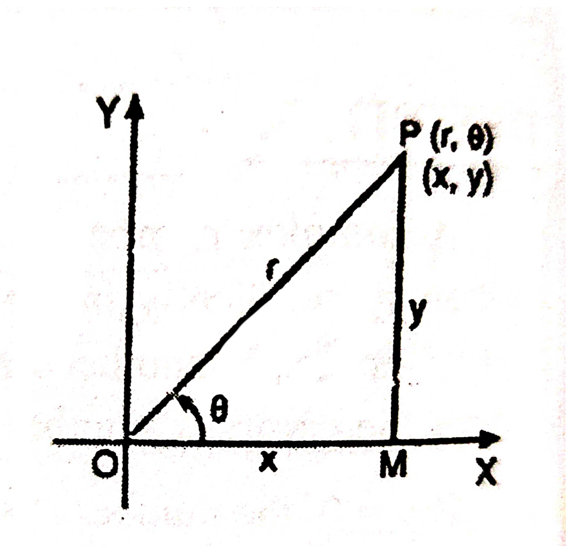
(vii) $|z| = |\overline{z}|$

(viii) $|z_1.z_2| = |z_1|.|z_2|$

(ix) $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}, z_2 \neq 0.$

1.2 Polar form of a Complex Number

Let (r, θ) be the polar coordinates of a point P, w.r.t. O as pole and OX, as the initial line. Thus $OP = r$ and $\angle XOP = \theta$. From P, draw $PM \perp OX$. let (x, y) be the cartesian coordinates of the point P, w.r.t. OX as x-axis and a line through O perpendicular to OX as y-axis. Then $x = rcos\ \theta$ and $y = rsin\ \theta$. Squaring and adding both we get $r = \sqrt{x^2 + y^2}$ and dividing both we get, $tan\ \theta = \frac{y}{x}$ which implies $\theta = tan^{-1}(\frac{y}{x})$.



$\therefore$ the point P represents the complex number as $z = x + iy = rcos\ \theta + irsin\ \theta = r(cos\ \theta +$

$isin\ \theta)$. This form is known as polar form of a complex number.

Note: 1. $r = \sqrt{x^2 + y^2}$ is called the modulus of the complex number $z = x + iy$ and is written as $r = |x + iy|$.
2. The number θ is called the amplitude or argument of the complex number.
3. The form $z = r(cos\ \theta + isin\ \theta)$ is also called modulus amplitude form.
4. $cos\ \theta + isin\ \theta$ is generally written as $cis\ \theta$
5. For finding the amplitude of a complex number we find the value of $\alpha = tan^{-1}|\frac{y}{x}|$.

a) If the complex number lie in the first quadrant, then $\theta = \alpha$

b) If complex number lie in the second quadrant, then $\theta = \pi - \alpha$

c) If complex number in the third quadrant, then $\theta = \alpha - \pi$

d) If complex number lie in the fourth quadrant, then $\theta = -\alpha$ or $2\pi - \alpha$

Example 1.2.1. *Represent the complex number $z = 1 + i\sqrt{3}$ in the polar form.*

Solution: Let $1 + i\sqrt{3} = r(cos\ \theta + isin\ \theta)$

$\therefore rcos\ \theta = 1$ $\cdots (1)$

and $rsin\ \theta = \sqrt{3}$ $\cdots (2)$

Squaring and adding (1) and (2), we get

$$r^2(cos^2\theta + sin^2\theta) = 1 + 3$$

$\implies r^2 = 4 \implies r = 2$

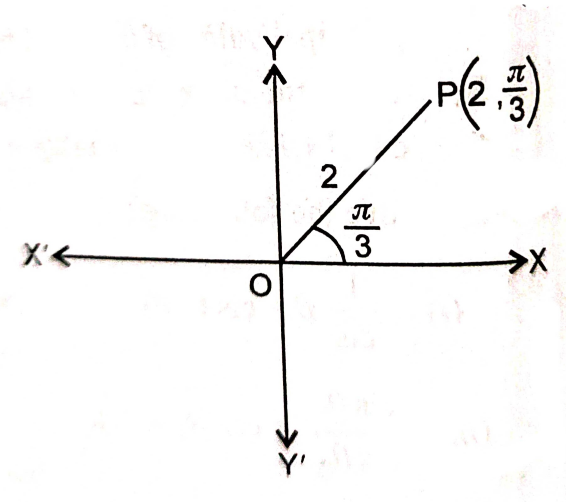
$\therefore$ from (1) and (2),

$$cos\ \theta = \frac{1}{2}, \quad sin\ \theta = \frac{\sqrt{3}}{2}$$

$\therefore \theta = \frac{\pi}{3}$

$\therefore z = 1 + i\sqrt{3} = r(cos\frac{\pi}{3} + isin\frac{\pi}{3})$

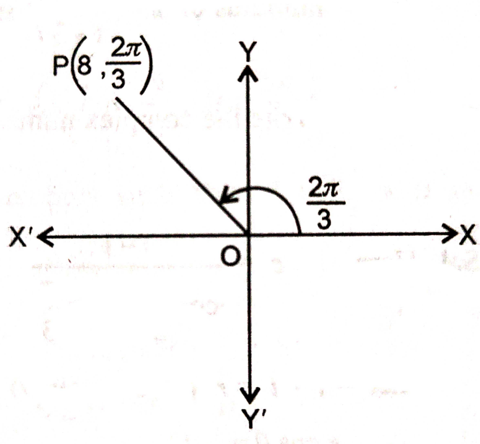
Thus complex number $z = 1+i\sqrt{3}$ is represented by $P(2, \frac{\pi}{3})$ in polar form.



Example 1.2.2. *Convert the complex number* $\dfrac{-16}{1+i\sqrt{3}}$ *into polar form.*

Solution: For converting the complex number $z = \dfrac{-16}{1+i\sqrt{3}}$ into polar form, we first convert it into standard $a + ib$ form by rationalizing the denominator.

$$z = \frac{-16}{1+i\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$
$$= \frac{-16(1-i\sqrt{3})}{1-i^2 3} = \frac{-16(1-i\sqrt{3})}{1+3}$$
$$= \frac{-16(1-i\sqrt{3})}{4} = -4(1-i\sqrt{3})$$
$$= 4 - i4\sqrt{3}$$



Let $4 - i4\sqrt{3} = r(cos\ \theta + isin\ \theta)$

$\therefore rcos\ \theta = -4 \cdots (1)$

and $rsin\ \theta = 4\sqrt{3} \cdots (2)$

Squaring and adding (1) and (2), we get

$$r^2(cos^2\theta + sin^2\theta) = 16 + 48$$

$\implies r^2 = 64 \implies r = 8$
$\therefore$ from (1) and (2),

$$cos\ \theta = -\frac{1}{2}, sin\ \theta = \frac{\sqrt{3}}{2}$$

$\therefore$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$\therefore$

$$z = 8(cos\frac{2\pi}{3} + isin\frac{2\pi}{3})$$

Hence z is represented by $P(8, \frac{2\pi}{3})$.

Example 1.2.3. *Write the complex number* $z = \dfrac{i-1}{cos\frac{\pi}{3} + isin\frac{\pi}{3}}$ *in polar form.*

Solution: Here $z = \dfrac{i-1}{cos\frac{\pi}{3} + isin\frac{\pi}{3}}$ $\cdots(1)$
Let $-1 + i = r(cos\ \theta + isin\ \theta)$
$\therefore rcos\ \theta = -1$ $\cdots(2)$
and $rsin\ \theta = 1$ $\cdots(3)$
Squaring and adding (2) and (3), we get,

$$r^2(cos^2\theta + sin^2\theta) = 1 + 1$$

or $r^2 = 2 \implies r = \sqrt{2}$
From (2) and (3), $cos\ \theta = -\frac{1}{\sqrt{2}}, sin\ \theta = \frac{1}{\sqrt{2}}$

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$\therefore \theta = \frac{3\pi}{4}$ as θ lies in the second quadrant
$\therefore -1 + i = \sqrt{2}(cos\ \frac{3\pi}{4} + isin\frac{3\pi}{4})$
$\therefore$ from (1), we get,

$$z = \frac{\sqrt{2}\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)}{cos\frac{\pi}{3} + isin\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)}{cos\frac{\pi}{3} + isin\frac{\pi}{3}} \times \frac{cos\frac{\pi}{3} - isin\frac{\pi}{3}}{cos\frac{\pi}{3} - isin\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left(cos\frac{3\pi}{4}cos\frac{\pi}{3} - icos\frac{3\pi}{4}sin\frac{\pi}{3} + isin\frac{3\pi}{4}cos\frac{\pi}{3} - i^2 sin\frac{3\pi}{4}sin\frac{\pi}{3}\right)}{cos^2\frac{\pi}{3} - i^2 sin^2\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left[\left(cos\frac{3\pi}{4}cos\frac{\pi}{3} + sin\frac{3\pi}{4}sin\frac{\pi}{3}\right) + i\left(sin\frac{3\pi}{4}cos\frac{\pi}{3} - cos\frac{3\pi}{4}sin\frac{\pi}{3}\right)\right]}{cos^2\frac{\pi}{3} + sin^2\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left[cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + isin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]}{1}$$

$$= \sqrt{2}\left(cos\frac{5\pi}{12} + isin\frac{5\pi}{12}\right)$$

1.3 De-Moivre's Theorem

Abraham de Moivre (1667-1754) was one of the mathematicians to use complex numbers in trigonometry. The formula

$$(cos\ \theta + isin\ \theta)^n = (cos\ n\theta + isin\ n\theta)$$

known by his name, was instrumental in bringing trigonometry out of the realm of geometry and into that of analysis.


de Moivre

Theorem 1.3.1. ***De-Moivre's Theorem***
(i) If n is any integer, then $(cos\ \theta + isin\ \theta)^n = (cos\ n\theta + isin\ n\theta)$ *and (ii) if n is a fraction, then one of the values of* $(cos\ \theta + isin\ \theta)^n$ *is* $(cos\ n\theta + isin\ n\theta)$*.*

Proof. **(i)** We will prove the theorem in three cases:
Case 1: When n is a positive integer. We prove the result by principle of mathematical induction
For n = 1,

$$(cos\theta + isin\theta)^n = (cos\theta + isin\theta)^1 = cos\theta + isin\theta$$

$\therefore$ result is true for n = 1.
Assume that the result is true for n == k, i.e.,

$$(cos\theta + isin\theta)^k = (cosk\theta + isink\theta)$$

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We prove the result for n = k+1, i.e.,

$$(cos\theta + isin\theta)^{k+1} = (cos(k+1)\theta + isin(k+1)\theta)$$

Now

$$\begin{aligned}(cos\theta + isin\theta)^{k+1} &= (cos\theta + isin\theta)^k (cos\theta + isin\theta)^1 \\ &= (cosk\theta + isink\theta)(cos\theta + isin\theta) \\ &= cos(k\theta + \theta) + isin(k\theta + \theta) \quad [\therefore cis\theta_1.cis\theta_2 = cis(\theta_1 + \theta_2)] \\ &= cos(k+1)\theta + isin(k+1)\theta\end{aligned}$$

$\therefore$ the result is true for n = k + 1.
Hence it is true for all positive integers.
Case II: When n is negative integer.
Let $n = -m$, where m is a +ive integer.
$\therefore$

$$\begin{aligned}(cos\theta + isin\theta)^n &= (cos\theta + isin\theta)^{-m} \\ &= \frac{1}{(cos\theta + isin\theta)^m} = \frac{1}{cosm\theta + isinm\theta} \\ &= \frac{1}{cosm\theta + isinm\theta} \times \frac{cosm\theta - isinm\theta}{cosm\theta - isinm\theta} \\ &= \frac{cosm\theta - isinm\theta}{cos^2m\theta - i^2sin^2m\theta} \\ &= \frac{cosm\theta - isinm\theta}{cos^2m\theta + sin^2m\theta} \\ &= cosm\theta - isinm\theta \\ &= cos(-m\theta) + isin(-m\theta) \\ &= cosn\theta + isinn\theta\end{aligned}$$

Hence the result is true for -ive integers also.

Case III: When n = 0.

$(cos\theta + isin\theta)^0 = 1 = cos0.\theta + isin0.\theta$

$\therefore$ (i) is true for all integers.

(ii): When n is a fraction

Let $n = \dfrac{p}{q}, q \neq 0$ where q is a +ve integer and p is any integer, $(p, q) = 1$

$\therefore$ from case 1, $(cos\dfrac{\theta}{q} + isin\dfrac{\theta}{q})^q = cos(q.\frac{\theta}{q}) + isin(q.\frac{\theta}{q}) = cos\theta + isin\theta$

$\therefore$ taking qth roots of both sides, $cos\frac{\theta}{q} + isin\frac{\theta}{q}$ is one of the values of $(cos\theta + isin\theta)^{\frac{1}{q}}$

$\therefore (cos\dfrac{\theta}{q} + isin\dfrac{\theta}{q})^p$ is one of the values of $(cos\theta + isin\theta)^{\frac{p}{q}}$

or $cos\frac{p}{q}\theta + isin\frac{p}{q}\theta$ is one of the value of $(cos\theta + isin\theta)^{\frac{p}{q}}$

or $cosn\theta + isinn\theta$ is one of the value of $(cos\theta + isin\theta)^n$.

■

Cor. 1: $(cos\theta - isin\theta)^n = [cos(-\theta) + isin(-\theta)]^n = cos(-n\theta) + isin(-n\theta) = cosn\theta - isinn\theta$

Cor. 2: $(cos\theta + isin\theta)^{-n} = cos(-n\theta) + isin(-n\theta) = cosn\theta - isinn\theta$

Cor. 3: $(cos\theta - isin\theta)^{-n} = [cos(-\theta) + isin(-\theta)]^{-n} = cosn\theta + isinn\theta$

Cor. 4: $\dfrac{1}{cos\theta + isin\theta} = (cos\theta + isin\theta)^{-1} = cos\theta - isin\theta$

1.4 Examples on De-Moivre's Theorem

Example 1.4.1. *Prove that* $(\frac{cos\theta + isin\theta}{sin\theta + icos\theta})^4 = cos8\theta + isin8\theta$

Proof: L.H.S.

$$= (\frac{cos\theta + isin\theta}{sin\theta + icos\theta})^4 = \frac{(cos\theta + isin\theta)^4}{\left[cos(\frac{\pi}{2} - \theta) + isin(\frac{\pi}{2} - \theta)\right]^4}$$
$$= \frac{cos4\theta + isin4\theta}{cos(2\pi - \theta) + isin(2\pi - \theta)}$$
$$= \frac{cos4\theta + isin4\theta}{cos4\theta - isin4\theta}$$
$$= (cos4\theta + isin4\theta)(cos4\theta - isin4\theta)^{-1}$$
$$= (cos4\theta + isin4\theta)(cos4\theta + isin4\theta)$$
$$= (cos8\theta + isin8\theta)$$

Example 1.4.2. *Simplify* $\left(sin\frac{\pi}{6} + icos\frac{\pi}{6}\right)^{18}$

Sol: We have,

$$\left(sin\frac{\pi}{6} + icos\frac{\pi}{6}\right)^{18} = \left[i\left(cos\frac{\pi}{6} - isin\frac{\pi}{6}\right)\right]^{18}$$
$$= i^{18}\left(cos\frac{\pi}{6} - isin\frac{\pi}{6}\right)^{18}$$
$$= (i^2)^9\left(cos\frac{18\pi}{6} - isin\frac{18\pi}{6}\right)$$
$$= (-1)^9(cos3\pi - isin3\pi)$$
$$= (-1)(-1 - i0)$$
$$= 1$$

Example 1.4.3. *Simplify* $\left(\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right)^{30}$

Sol: Let $z = 1 + cos2\theta + isin2\theta$
As $|z| = |z|^2 = z\overline{z} = 1$, we get $\overline{z} = \frac{1}{z} = cos2\theta - isin2\theta$
Therefore, $\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta} = \frac{1+z}{1+\frac{1}{z}} = z$
Therefore,

$$\begin{aligned}\left(\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right)^{30} &= z^{30}\\ &= (cos2\theta + isin2\theta)^{30}\\ &= cos60\theta + isin60\theta\end{aligned}$$

Example 1.4.4. *Simplify* $(1+i)^{18}$

Sol: Let $1 + i = r(cos\theta + isin\theta)$. Then we get,

$$r\sqrt{1^2+1^2} = \sqrt{2}, \alpha = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4},$$

$$\theta = \alpha = \frac{\pi}{4}$$

Therefore $1 + i = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$
Raising to power 18 both sides
$(1+i)^{18} = \left[\sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)\right]^{18} = \sqrt{2}^{18}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)^{18}$

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Using De-Moivre's theorem

$$
\begin{aligned}
(1+i)^{18} &= 2^9\left(cos\frac{18\pi}{4} + isin\frac{18\pi}{4}\right)\\
&= 2^9\left(cos(4\pi+\frac{\pi}{2}) + isin(4\pi+\frac{\pi}{2})\right)\\
&= 2^9\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)\\
&= 2^9(0+1i)\\
&= 512i
\end{aligned}
$$

Example 1.4.5. *Prove that*

$$(1+sin\theta+icos\theta)^n+(1+sin\theta-icos\theta)^n = 2^{n+1}cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right).$$

Sol: L.H.S.

$$
\begin{aligned}
&= (1+sin\theta+icos\theta)^n + (1+sin\theta-icos\theta)^n\\
&= \left(1+cos\left(\frac{\pi}{2}-\theta\right)+isin\left(\frac{\pi}{2}-\theta\right)\right)^n + \left(1+cos\left(\frac{\pi}{2}-\theta\right)-isin\left(\frac{\pi}{2}-\theta\right)\right)^n\\
&= \left[2cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i2sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\\
&+ \left[2cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i2sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\\
&= \left[2cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\left[cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+isin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\\
&+ \left[2cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\left[cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-isin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\\
&= \left[2cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^n\\
&\left[cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)+isin\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)+cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)-isin\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&= 2^n cos^n \left(\frac{\pi}{4} - \frac{\theta}{2}\right) \left[2cos\left(\frac{n\pi}{4} - \frac{n\theta}{2}\right)\right] \\
&= 2^{n+1} cos^n \left(\frac{\pi}{4} - \frac{\theta}{2}\right) 2cos\left(\frac{n\pi}{4} - \frac{n\theta}{2}\right) \\
&= R.H.S.
\end{aligned}
$$

Example 1.4.6. *Evaluate* $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{1000}$.

Sol: Let $z = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = r(cos\theta + isin\theta)$
Therefore, $rcos\theta = \frac{\sqrt{2}}{2}$
and $rsin\theta = \frac{\sqrt{2}}{2}$
Squaring and adding, we get

$$r^2cos^2\theta + r^2sin^2\theta = (\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = 1$$

and

$$tan\theta = 1 \implies \theta = \frac{\pi}{4}$$

Therefore, $z = 1(cos\frac{\pi}{4} + isin\frac{\pi}{4})$

$$
\begin{aligned}
\implies z^{1000} &= \left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)^{1000} \\
&= cos\frac{1000\pi}{4} + isin\frac{1000\pi}{4} \\
&= cos\ 250\pi + isin\ 250\pi \\
&= cos(0 + 125 \times 2\pi) + isin(0 + 125 \times 2\pi) \\
&= 1
\end{aligned}
$$

$\therefore \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^{1000} = 1.$

Example 1.4.7. *Find all the values of* $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ *and show that the* *continued product of all the values is 1.*

Sol:

Let $\frac{1}{2}+i\frac{\sqrt{3}}{2}=r(cos\theta+isin\theta)$
Equating real and imaginary parts,

$$rcos\theta=\frac{1}{2}$$

$$rsin\theta=\frac{\sqrt{3}}{2}$$

Squaring and adding, we get

$$r^2=\frac{1}{4}+\frac{3}{4},$$

$\therefore r=1.$
Also, $cos\theta=\frac{1}{2}, sin\theta=\frac{\sqrt{3}}{2} \implies \theta=\frac{\pi}{3}$

$$\begin{aligned}\therefore \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} &= \left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^{\frac{3}{4}} = \left[\left(cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}\\ &= (cos\pi+isin\pi)^{\frac{1}{4}}\\ &= (cos(2n\pi+\pi)+isin(2n\pi+\pi))^{\frac{1}{4}}\\ &= (cos(2n+1)\pi+isin(2n+1)\pi)^{\frac{1}{4}}\\ &= cos\frac{(2n+1)\pi}{4}+isin\frac{(2n+1)\pi}{4} \quad \text{where} \quad n=0,1,2,3.\end{aligned}$$

Putting $n=0,1,2,3$, the values are

$$cos\frac{\pi}{4}+isin\frac{\pi}{4}, cos\frac{3\pi}{4}+isin\frac{3\pi}{4}, cos\frac{5\pi}{4}+isin\frac{5\pi}{4}, cos\frac{7\pi}{4}+isin\frac{7\pi}{4}$$

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Continued product of all these values

$$= \left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)\left(cos\frac{3\pi}{4} + isin\frac{3\pi}{4}\right)\left(cos\frac{5\pi}{4} + isin\frac{5\pi}{4}\right)\left(cos\frac{7\pi}{4} + isin\frac{7\pi}{4}\right)$$
$$= cis\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$
$$= cis4\pi$$
$$= cos4\pi + isin4\pi$$
$$= (cos\pi + isin\pi)^4$$
$$= (-1)^4$$
$$= 1$$

Exercise: If $a = cos\alpha + isin\alpha, b = cos\beta + isin\beta, c = cos\gamma + isin\gamma$ prove that

(i) $\frac{ab}{c} + \frac{c}{ab} = 2cos(\alpha + \beta - \gamma)$

(ii) $abc + \frac{1}{abc} = 2cos(\alpha + \beta + \gamma)$

(iii) $a^pb^qc^r + \frac{1}{a^pb^qc^r} = 2cos(p\alpha + q\beta + r\gamma)$

Solution:

(i)

$$\frac{ab}{c} + \frac{c}{ab} = \frac{(cos\alpha + isin\alpha)(cos\beta + isin\beta)}{cos\gamma + isin\gamma} + \frac{cos\gamma + isin\gamma}{(cos\alpha + isin\alpha)(cos\beta + isin\beta)}$$
$$= \frac{cos(\alpha+\beta) + isin(\alpha+\beta)}{cos\gamma + isin\gamma} + \frac{cos\gamma + isin\gamma}{cos(\alpha+\beta) + isin(\alpha+\beta)}$$
$$= (cos(\alpha+\beta) + isin(\alpha+\beta))(cos\gamma + isin\gamma)^{-1}$$
$$+ (cos\gamma + isin\gamma)(cos(\alpha+\beta) + isin(\alpha+\beta))^{-1}$$

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$= (cos(\alpha+\beta) + isin(\alpha+\beta))(cos\gamma - isin\gamma)$
$+ (cos\gamma + isin\gamma)(cos(\alpha+\beta) - isin(\alpha+\beta))$
$= (cos(\alpha+\beta) + isin(\alpha+\beta))(cos(-\gamma) + isin(-\gamma))$
$+ (cos\gamma + isin\gamma)(cos(-(\alpha+\beta)) + isin(-(\alpha+\beta)))$
$= (cos(\alpha+\beta-\gamma) + isin(\alpha+\beta-\gamma)) + (cos(\gamma-\alpha-\beta) + isin(\gamma-\alpha-\beta))$
$= (cos(\alpha+\beta-\gamma) + isin(\alpha+\beta-\gamma)) + (cos(-(\alpha+\beta-\gamma)) + isin(-(\alpha+\beta-\gamma)))$
$= (cos(\alpha+\beta-\gamma) + isin(\alpha+\beta-\gamma)) + (cos(\alpha+\beta-\gamma) - isin(\alpha+\beta-\gamma))$
$= 2cos(\alpha+\beta-\gamma)$

Similarly we can prove (ii) and (iii).

Example 1.4.8. *If* $cos\alpha + cos\beta + cos\gamma = sin\alpha + sin\beta + sin\gamma = 0$, *prove that*
(i) $cos3\alpha + cos3\beta + cos3\gamma = 3cos(\alpha+\beta+\gamma)$
(ii) $sin3\alpha + sin3\beta + sin3\gamma = 3sin(\alpha+\beta+\gamma)$

Sol: Let $a = cos\alpha + isin\alpha$, $b = cos\beta + isin\beta$, $c = cos\gamma + isin\gamma$
$\therefore a + b + c = (cos\alpha + cos\beta + cos\gamma) + i(sin\alpha + sin\beta + sin\gamma) =) + i0$
$\therefore a + b + c = 0$
$\implies a^3 + b^3 + c^3 = 3abc$
$\implies (cos\alpha + isin\alpha)^3 + (cos\beta + isin\beta)^3 + (cos\gamma + isin\gamma)^3 = 3(cos\alpha + isin\alpha)(cos\beta + isin\beta)(cos\gamma + isin\gamma)$
$\implies (cos3\alpha + isin3\alpha) + (cos3\beta + isin3\beta) + (cos3\gamma + isin3\gamma) = 3[cos(\alpha+\beta+\gamma) + isin(\alpha+\beta+\gamma)]$
$\implies (cos\alpha + cos\beta + cos\gamma) + i(sin\alpha + sin\beta + sin\gamma) = 3cos(\alpha+\beta+\gamma) + i3sin(\alpha+\beta+\gamma)$
Equating real and imaginary parts, we get

$cos3\alpha + cos3\beta + cos3\gamma = 3cos(\alpha + \beta + \gamma)$
and $sin3\alpha + sin3\beta + sin3\gamma = 3sin(\alpha + \beta + \gamma)$

1.5 Applications of de-Moivre's theorem

Finding n^{th} roots of Complex number

de-Moivre's formula can be used to obtain roots of complex numbers. Suppose n is a positive integer and a complex number ω is the n^{th} root of z denoted by $z^{\frac{1}{n}}$, then we have

$$\omega^n = z \qquad \cdots (1)$$

Let

$$\omega = \rho(cos\phi + isin\phi)$$

and

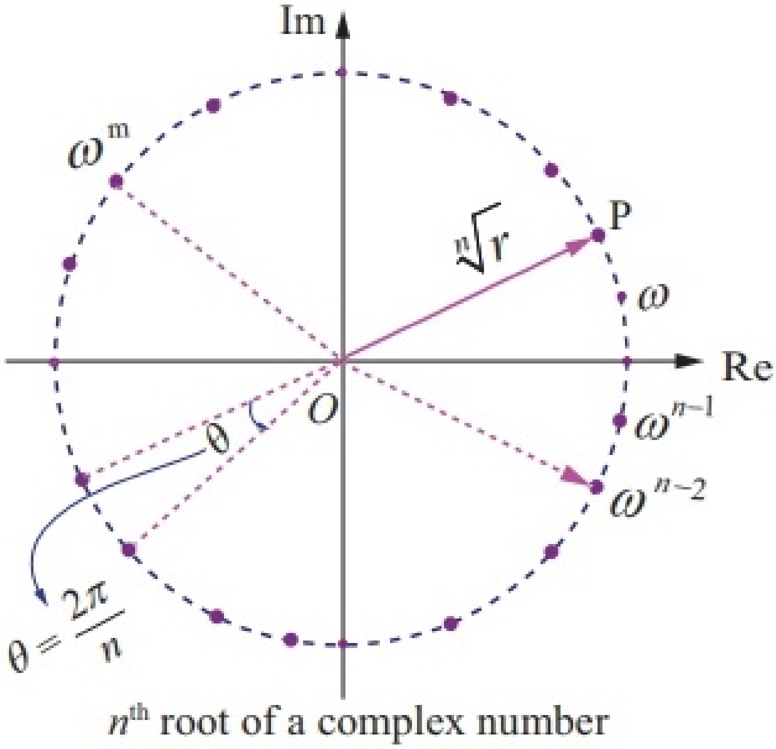
$$z = r(cos\theta + isin\theta) = r(cos(\theta + 2k\pi) + isin(\theta + 2k\pi)), k \in \mathbb{Z}$$

Since ω is the nth root of z, then $\omega^n = z$
$\implies \rho^n(cos\phi + isin\phi)^n = r(cos(\theta + 2k\pi) + isin(\theta + 2k\pi)), k \in \mathbb{Z}$
By de-Moivre's theorem,
$\implies \rho^n(cosn\phi + isinn\phi) = r(cos(\theta + 2k\pi) + isin(\theta + 2k\pi)), k \in \mathbb{Z}$
Comparing the moduli and arguments, we get
$\rho^n = r$ and $n\phi = \theta + 2k\pi, k \in \mathbb{Z}$
$\implies \rho = r^{\frac{1}{n}}$ and $\phi = \frac{\theta+2k\pi}{n}, k \in \mathbb{Z}$ Although there are infinitely many values of k, the distinct values of ω are obtained when $k = 0, 1, 2, ..., n-1$. When $k = n, n+1, n+2, ...$we get the same roots at regular intervals (cylically). Therefore the n roots of complex number $z = r(cos\theta + isin\theta)$

are

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(cos\left(\frac{\theta + 2k\pi}{n}\right) + isin\left(\frac{\theta + 2k\pi}{n}\right)\right), \ k = 0, 1, 2, ..., n-1.$$

If we set $\omega = \sqrt[n]{r}e^{\frac{i(\theta+2k\pi)}{n}}$ the formula for the nth roots of a complex number has a nice geometric interpretation, as shown in fig. Note that because $|\omega| = \sqrt[n]{r}$ the n roots all have the same modulus $\sqrt[n]{r}$ they all lie on a circle of radius $\sqrt[n]{r}$ with centre at the origin. Furthermore, the n roots are equally spaced along the circle, because successive n roots have arguments that differ by $\frac{2\pi}{n}$.



n^{th} root of a complex number

Find the nth root of Unity

The solutions of the equation$z^n = 1$, for positive values of integer n, are

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the nth roots of the unity. In polar form the equation $z^n = 1$ can be written as

$$z^n = cos(0 + 2k\pi) + isin(0 + 2k\pi) = e^{i2k\pi}, k = 0, 1, 2, ...$$

Using deMoivre's theorem, we find the nth roots of unity from the equation given below:

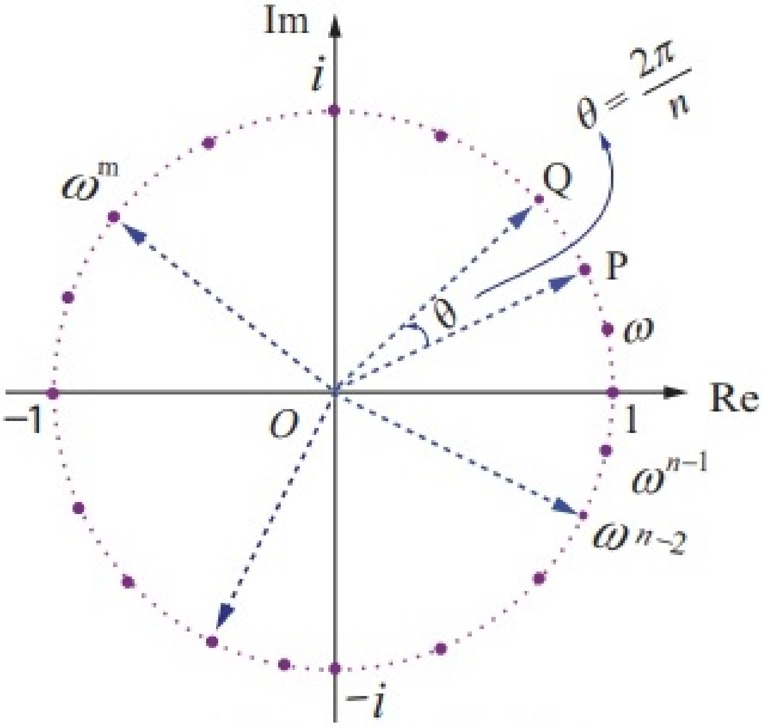
$$z = cos\left(\frac{2k\pi}{n}\right) + isin\left(\frac{2k\pi}{n}\right) = e^{\frac{i2k\pi}{n}}, k = 0, 1, 2, ..., n-1 \qquad \cdots (1)$$

Given a positive integer n, a complex number z is called an nth root of unity if and only if $z^n = 1$.
If we denote the complex number by ω, then

$$\omega = e^{\frac{2\pi i}{n}} = cos\frac{2\pi i}{n} + isin\frac{2\pi i}{n}$$

$$\implies \omega^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = 1.$$

Therefore ω is an nth root of unity. From equation (1), the complex numbers $1, \omega, \omega^2, ..., \omega^{n-1}$ are n roots of unity. The complex numbers $1, \omega, \omega^2, ..., \omega^{n-1}$ are the points in the complex plane and are the vertices of a regular polygon of n sides inscribed in a unit circle as shown in figure. Note that because the nth roots all have the same modulus 1, they will lie on a circle of radius 1 with centre at the origin. Furthermore, the n roots are equally spaced along the circle, because successive nth roots have arguments that differ by $\frac{2\pi}{n}$.
The nth roots of unity $1, \omega, \omega^2, ..., \omega^{n-1}$ are in geometric progression with common ratio ω.



n^{th} roots of unity

Therefore $1+\omega+\omega^2+...+\omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0$ since $\omega^n = 1$ and $\omega \neq 1$.
The product of n, nth roots of unity is

$$\begin{aligned} 1\ \omega\ \omega^2\ ...\ \omega^{n-1} &= \omega^{0+1+2+...+n-1} = \omega^{\frac{(n-1)n}{2}} \\ &= (\omega^n)^{\frac{n-1}{2}} = (e^{i2\pi})^{\frac{n-1}{2}} \\ &= (e^{i\pi})^{n-1} = (-1)^{n-1} \end{aligned}$$

$\therefore$ the product of all the nth roots of unity is $1\ \omega\ \omega^2\ ...\ \omega^{n-1} = (-1)^{n-1}$.
Since $|\omega| = 1$, we have $\omega\ \overline{\omega} = |\omega|^2 = 1$,
hence

$$\overline{\omega} = \omega^{-1} \implies (\overline{\omega})^k = \omega^{-k}, 0 \le k \le n-1$$

$$\omega^{n-k} = \omega^n\ \omega^{-k} = \omega^{-k} = (\overline{\omega})^k, 0 \le k \le n-1$$

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Therefore

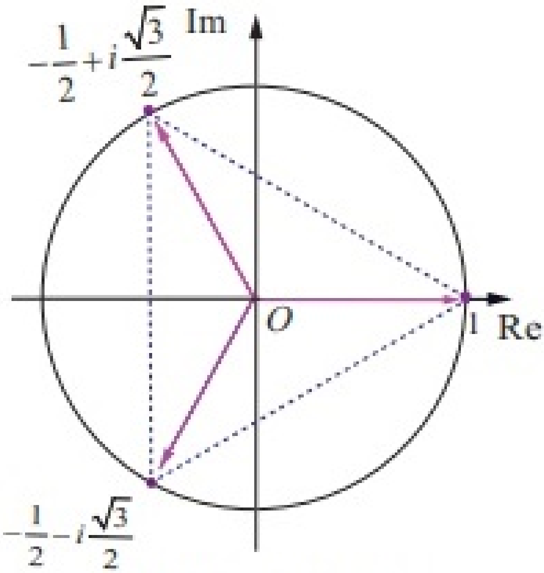
$$\omega^{n-k} = \omega^{-k} = (\overline{\omega})^k, 0 \leq k \leq n-1$$

Note:

1. All the n roots of nth roots of unity are in Geometric Progression.
2. Sum of the n roots of nth roots of unity is always equal to zero.
3. Product of the n roots of nth roots of unity is equal to $(-1)^{n-1}$.
4. All the n roots of nth roots of unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.

Example 1.5.1. *Find the cube roots of unity.*

Sol: We have to find $1^{\frac{1}{3}}$. Let $z = 1^{\frac{1}{3}}$, then $z^3 = 1$.
In polar form, the equation $z^3 = 1$ can be written as $z^3 = cos(0 + 2k\pi) + isin(0 + 2k\pi) = e^{i2k\pi}, k = 0, 1, 2, ...$



Cube roots of unity

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Therefore,

$$\begin{aligned} z &= cos\left(\frac{2k\pi}{3}\right) + isin\left(\frac{2k\pi}{3}\right) \\ &= e^{\frac{i2k\pi}{3}}, k = 0, 1, 2. \end{aligned}$$

Taking $k = 0, 1, 2$, we get

$$k = 0, \qquad z = cos\ 0 + isin\ 0 = 1$$

$$\begin{aligned} k = 1, \qquad z &= cos\frac{2\pi}{3} + isin\frac{2\pi}{3} \\ &= cos\left(\pi - \frac{\pi}{3}\right) + isin\left(\pi - \frac{\pi}{3}\right) \\ &= -cos\frac{\pi}{3} + isin\frac{\pi}{3} \\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ &= \frac{-1 + i\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} k = 2, \qquad z &= cos\frac{4\pi}{3} + isin\frac{4\pi}{3} \\ &= cos\left(\pi + \frac{\pi}{3}\right) + isin\left(\pi + \frac{\pi}{3}\right) \\ &= -cos\frac{\pi}{3} - isin\frac{\pi}{3} \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ &= \frac{-1 - i\sqrt{3}}{2} \end{aligned}$$

Therefore, the cube roots of unity are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2} \implies 1, \omega, \omega^2$, where $\omega = e^{\frac{i2\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$.

Finding solutions of equation

Example 1: Solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

Solution: The given equation is

$$x^4 - x^3 + x^2 - x + 1 = 0 \qquad \cdots (1)$$

Multiplying both sides by $x + 1$, we get

$$x^5 + 1 = 0 \qquad (2)$$

or

$$x^5 = -1$$

$$\begin{aligned} \therefore x &= (-1)^{\frac{1}{5}} = (cos\pi + isin\pi)^{\frac{1}{5}} \\ &= [cos(2n\pi + \pi) + isin(2n\pi + \pi)]^{\frac{1}{5}} \\ &= cos\frac{(2n+1)\pi}{5} + isin\frac{(2n+1)\pi}{5} \quad \text{where} \quad n = 0, 1, 2, 3, 4. \end{aligned}$$

Putting $n = 0, 1, 2, 3, 4$, the roots of (2) are

$cos\frac{\pi}{5} + isin\frac{\pi}{5}, cos\frac{3\pi}{5} + isin\frac{3\pi}{5}, cos\pi + isin\pi, cos\frac{7\pi}{5} + isin\frac{7\pi}{5}, cos\frac{9\pi}{5} + isin\frac{9\pi}{5}$

or $cos\frac{\pi}{5} + isin\frac{\pi}{5}, cos\frac{3\pi}{5} + isin\frac{3\pi}{5}, -1, cos\frac{3\pi}{5} - isin\frac{3\pi}{5}, cos\frac{\pi}{5} - isin\frac{\pi}{5}$

or $-1, cos\frac{r\pi}{5} \pm isin\frac{r\pi}{5}$ where $r = 1, 3$

But the root $= -1$ corresponds to the factor $x + 1$.

$\therefore$ roots of (1) are $cos\frac{r\pi}{5} \pm isin\frac{r\pi}{5}$ where $r = 1, 3$

Example 2: Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.

Solution: The given equation is $x^{12}-1=0 \qquad \cdots(1)$
or $x^{12}=1$ or $x=(1)^{\frac{1}{12}}$

$$\therefore x=(cos0+isin)^{\frac{1}{12}}=[cos(2n\pi+0)+isin(2n\pi+0)]^{\frac{1}{12}}$$
$$=cos\frac{n\pi}{6}+isin\frac{n\pi}{6} \text{ where } n=0,1,2,...,11.$$

Putting $n=0,1,2,...,11$, we get the roots as
$cos0+isin0$, $cos\frac{\pi}{6}+isin\frac{\pi}{6}$, $cos\frac{\pi}{3}+isin\frac{\pi}{3}$, $cos\frac{\pi}{2}+isin\frac{\pi}{2}$, $cos\frac{2\pi}{3}+isin\frac{2\pi}{3}$, $cos\frac{5\pi}{6}+isin\frac{5\pi}{6}$, $cos\pi+isin\pi$, $cos\frac{7\pi}{6}+isin\frac{7\pi}{6}$, $cos\frac{4\pi}{3}+isin\frac{4\pi}{3}$, $cos\frac{3\pi}{2}+isin\frac{3\pi}{2}$, $cos\frac{5\pi}{3}+isin\frac{5\pi}{3}$, $cos\frac{11\pi}{6}+isin\frac{11\pi}{6}$

or 1, $cos\frac{\pi}{6}+isin\frac{\pi}{6}$,$cos\frac{\pi}{3}+isin\frac{\pi}{3}$, i, $-cos\frac{\pi}{3}+isin\frac{\pi}{3}$, $-cos\frac{\pi}{6}+isin\frac{\pi}{6}$, -1, $-cos\frac{\pi}{6}-isin\frac{\pi}{6}$, $-cos\frac{\pi}{3}-isin\frac{\pi}{3}$, $-cos\frac{\pi}{2}-isin\frac{\pi}{2}$, $cos\frac{\pi}{3}-isin\frac{\pi}{3}$, $cos\frac{\pi}{6}-isin\frac{\pi}{6}$

or 1, $\frac{\sqrt{3}}{2}+i.\frac{1}{2}$, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, i, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}+i\frac{1}{2}$, -1, $-\frac{\sqrt{3}}{2}-i\frac{1}{2}$, $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $-i$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}-i\frac{1}{2}$

or ±1, $\pm i$, $\pm\frac{\sqrt{3}\pm i}{2}$, $\pm\frac{1\pm i\sqrt{3}}{2}$
Now consider $x^4+x^2+1=0$
Multiplying both sides by x^2-1, we get,
$(x^2-1)(x^4+x^2+1)=0$, or $x^6-1=0$
$\therefore x^6=1$ or $x=(1)^{\frac{1}{6}}=(cos0+isin0)^{\frac{1}{6}}$
$\therefore\ x=[cos(2n\pi+0)+isin(2n\pi+0)]^{\frac{1}{6}}=cos\frac{n\pi}{3}+isin\frac{n\pi}{3}$ where $n=0,1,2,3,4,5$.
Putting $n=0,1,2,3,4,5$, we get,
$cos0+isin0$, $cos\frac{\pi}{3}+isin\frac{\pi}{3}$, $cos\frac{2\pi}{3}+isin\frac{2\pi}{3}$, $cos\pi+isin\pi$, $cos\frac{4\pi}{3}+isin\frac{4\pi}{3}$, $cos\frac{5\pi}{3}+isin\frac{5\pi}{3}$

or $1+i.0,\ cos\frac{\pi}{3}+isin\frac{\pi}{3},\ -cos\frac{\pi}{3}+isin\frac{\pi}{3},\ -1+i.0,\ -cos\frac{\pi}{3}-isin\frac{\pi}{3},$ $cos\frac{\pi}{3}-isin\frac{\pi}{3}$

or $1,\ \frac{1}{2}+i\frac{\sqrt{3}}{2},\ -\frac{1}{2}+i\frac{\sqrt{3}}{2},\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2},\ \frac{1}{2}-i\frac{\sqrt{3}}{2}$

or $\pm1,\ \frac{1\pm i\sqrt{3}}{2},\ \frac{-1\pm i\sqrt{3}}{2}$

Rejecting roots $x=\pm1$ which are produced by multiplication of x^2-1, we get the roots as

$$\frac{1\pm i\sqrt{3}}{2},\frac{-1\pm i\sqrt{3}}{2}$$

These are also the roots of (1).

Example 3 : Prove that the roots of the equation $(x-1)^n=x^n$ (n being +ve integer) are $\frac{1}{2}\left[1+icot\frac{r\pi}{n}\right]$ where r has the values $0,1,2,...,n-1$.

Solution: The given equation is

$$(x-1)^n=x^n \quad \text{or} \quad \left(\frac{x-1}{x}\right)^n=1 \ \text{ or } \ \frac{x-1}{x}=(1)^{\frac{1}{n}}$$

$$\therefore \frac{x-1}{x}=(cos0+isin0)^{\frac{1}{n}}=[cos(2r\pi+0)+isin(2r\pi+0)]^{\frac{1}{n}}$$
$$=cos\frac{2r\pi}{n}+isin\frac{2r\pi}{n}, \quad \text{where} \quad r=0,1,2,...,n-1$$
$$\therefore \frac{x-1}{x}=cos\theta+isin\theta \quad \text{where}\theta=\frac{2r\pi}{n}$$
$$\Longrightarrow \frac{x}{x}-\frac{1}{x}=cos\theta+isin\theta$$
$$\text{or} \quad 1-\frac{1}{x}=cos\theta+isin\theta$$
$$\Longrightarrow \frac{1}{x}=1-cos\theta-isin\theta$$

$$\implies x = \frac{1}{1 - cos\theta - isin\theta} = \frac{1}{2sin^2\frac{\theta}{2} - 2isin\frac{\theta}{2}cos\frac{\theta}{2}}$$
$$= \frac{1}{-2isin\frac{\theta}{2}\left(cos\frac{\theta}{2} - \frac{1}{i}sin\frac{\theta}{2}\right)} = \frac{i}{2sin\frac{\theta}{2}}\left(cos\frac{\theta}{2} + isin\frac{\theta}{2}\right)^{-1}$$
$$= \frac{i}{2sin\frac{\theta}{2}}\left(cos\frac{\theta}{2} - isin\frac{\theta}{2}\right) = \frac{1}{2}\left[icot\frac{\theta}{2} + 1\right]$$
$$\therefore x = \frac{1}{2}\left[1 + icot\frac{r\pi}{n}\right], \quad \text{where} \quad r = 0, 1, 2..., n-1.$$

Exercise : Prove that the roots of the equation $(5+x)^5 - (5-x)^5 = 0$ are given by $x = 5itan\frac{r\pi}{5}$ where $r = 0, 1, 2, 3, 4$.

Solution: The given equation is

$$(5+x)^5 - (5-x)^5 = 0$$
$$(5+x)^5 = (5-x)^5$$
$$\left(\frac{5+x}{5-x}\right)^5 = 1$$
$$\frac{5+x}{5-x} = (1)^{\frac{1}{5}}$$
$$= (cos0 + isin0)^{\frac{1}{5}}$$
$$\frac{5+x}{5-x} = (cos(2r\pi + 0) + isin(2r\pi + 0))^{\frac{1}{5}} \quad \text{where} \quad n = 0, 1, 2, 3, 4.$$
$$= cos\frac{2r\pi}{5} + isin\frac{2r\pi}{5}$$
$$= \frac{cos\theta + isin\theta}{1} \quad \text{where} \quad \theta = \frac{2r\pi}{5}$$

Applying Componendo and Dividendo, we get

$$\begin{aligned}
\frac{(5+x)+(5-x)}{(5+x)-(5-x)} &= \frac{cos\theta + isin\theta + 1}{cos\theta + isin\theta - 1}\\
\frac{10}{2x} &= \frac{1 + cos\theta + isin\theta}{-(1-cos\theta) + isin\theta}\\
\frac{5}{x} &= \frac{2cos^2\frac{\theta}{2} + 2isin\frac{\theta}{2}cos\frac{\theta}{2}}{-2sin^2\frac{\theta}{2} + 2isin\frac{\theta}{2}cos\frac{\theta}{2}}\\
\frac{5}{x} &= \frac{2cos\frac{\theta}{2}\left(cos\frac{\theta}{2} + isin\frac{\theta}{2}\right)}{2isin\frac{\theta}{2}\left(cos\frac{\theta}{2} - \frac{1}{i}sin\frac{\theta}{2}\right)}\\
&= \frac{1}{i}cot\frac{\theta}{2}\\
\therefore \frac{5}{x} &= \frac{1}{i}cot\frac{\theta}{2}\\
\implies \frac{x}{5} &= \frac{i}{cot\frac{\theta}{2}}\\
&= itan\frac{\theta}{2}\\
x &= 5itan\frac{\theta}{2}\\
&= 5itan\frac{2r\pi}{10}\\
&= 5itan\frac{r\pi}{5} \quad \text{where} \quad r = 0, 1, 2, 3, 4.
\end{aligned}$$

Exercise: Solve the equation $(1+x)^3 = i(1-x)^3$

Solution: The given equation

$$\begin{aligned}
(1+x)^3 &= i(1-x)^3\\
\left(\frac{1+x}{1-x}\right)^3 &= i\\
\frac{1+x}{1-x} &= (i)^{\frac{1}{3}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)^{\frac{1}{3}} \\
&= \left(cos(2n\pi + \frac{\pi}{2}) + isin(2n\pi + \frac{\pi}{2})\right)^{\frac{1}{3}} \\
&= \left(cos(4n+1)\frac{\pi}{2} + isin(4n+1)\frac{\pi}{2}\right)^{\frac{1}{3}} \\
&= cos(4n+1)\frac{\pi}{6} + isin(4n+1)\frac{\pi}{6} \\
&= \frac{cos\theta + isin\theta}{1} \quad \text{where} \quad \theta = \frac{(4n+1)\pi}{6}
\end{aligned}
$$

Applying Componendo and Dividendo, we get

$$
\begin{aligned}
\frac{(1+x)+(1-x)}{(1+x)-(1-x)} &= \frac{cos\theta + isin\theta + 1}{cos\theta + isin\theta - 1} \\
\frac{2}{2x} &= \frac{1 + cos\theta + isin\theta}{-(1-cos\theta) + isin\theta} \\
\frac{1}{x} &= \frac{2cos^2\frac{\theta}{2} + 2isin\frac{\theta}{2}cos\frac{\theta}{2}}{-2sin^2\frac{\theta}{2} + 2isin\frac{\theta}{2}cos\frac{\theta}{2}} \\
\frac{1}{x} &= \frac{2cos\frac{\theta}{2}\left(cos\frac{\theta}{2} + isin\frac{\theta}{2}\right)}{2isin\frac{\theta}{2}\left(cos\frac{\theta}{2} - \frac{1}{i}sin\frac{\theta}{2}\right)} \\
&= \frac{1}{i}cot\frac{\theta}{2} \\
\therefore \frac{1}{x} &= \frac{1}{i}cot\frac{\theta}{2} \\
\implies x &= \frac{i}{cot\frac{\theta}{2}} \\
&= itan\frac{\theta}{2} \\
x &= itan\frac{\theta}{2} \\
&= itan\frac{(4n+1)\pi}{12}, \quad \text{where} \quad n = 0, 1, 2.
\end{aligned}
$$

Exercise: Solve $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

Solution: The given equation is $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$.
Multiplying b/s by $z - 1$, we get

$$\begin{aligned} z^6 - 1 &= 0 \\ z^6 &= 1 \\ z &= (1)^{\frac{1}{6}} \\ &= (cos0 + isin0)^{\frac{1}{6}} \\ &= (cos(2n\pi + 0) + isin(2n\pi + 0))^{\frac{1}{6}} \\ &= cos\frac{2n\pi}{6} + isin\frac{2n\pi}{6} \\ &= cos\frac{n\pi}{3} + isin\frac{n\pi}{3}, \quad \text{where} \quad n = 0, 1, 2, 3, 4, 5. \end{aligned}$$

Neglecting the root for $n = 0$ as induced by the multiplication by $z - 1$,
$\therefore$ the roots of the equation are given by

$$z = cos\frac{n\pi}{3} + isin\frac{n\pi}{3}, \quad \text{where} \quad n = 1, 2, 3, 4, 5.$$

Expansion of $cos^n\theta$ in terms of cosines of multiples of θ

Let $x = cos\theta + isin\theta, \quad \therefore x^n = cosn\theta + isinn\theta$
$\frac{1}{x} = cos\theta - isin\theta, \quad \frac{1}{x^n} = cosn\theta - isinn\theta$
$\therefore x + \frac{1}{x} = 2cos\theta, x^n + \frac{1}{x^n} = 2cosn\theta$
Now $(2cos\theta)^n = \left(x + \frac{1}{x}\right)^n$
$= {}^nC_0x^n + {}^nC_1x^{n-1}.\frac{1}{x} + {}^nC_2x^{n-2}.\frac{1}{x^2} + ... + {}^nC_{n-2}x^2.\frac{1}{x^{n-2}} + {}^nC_{n-1}x.\frac{1}{x^{n-1}} + {}^nC_n\frac{1}{x^n}$

$= {}^nC_0x^n + {}^nC_1x^{n-2} + {}^nC_2x^{n-4} + ... + {}^nC_2\frac{1}{x^{n-4}} + {}^nC_1\frac{1}{x^{n-2}} + {}^nC_0\frac{1}{x^n}$

Two cases arise:

Case 1: n is even

In this case, there is only one middle term

$$T_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}} x^{n-\frac{n}{2}} \frac{1}{x^{\frac{n}{2}}} = {}^nC_{\frac{n}{2}}$$

$$\therefore \quad (2cos\theta)^n = {}^nC_0 \left(x^n + \frac{1}{x^n}\right) + {}^nC_1 \left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2 \left(x^{n-4} + \frac{1}{x^{n-4}}\right) + ... + {}^nC_{\frac{n}{2}}$$

$$\therefore \quad 2^n cos^n\theta = {}^nC_0.(2cosn\theta) + {}^nC_1.\{2cos(n-2)\theta\} + {}^nC_2.\{2cos(n-4)\theta\} + ... + {}^nC_{\frac{n}{2}}$$

$$\therefore \quad cos^n\theta = \frac{1}{2^{n-1}}[{}^nC_0.\{cosn\theta\} + {}^nC_1.\{cos(n-2)\theta\} + {}^nC_2.\{cos(n-4)\theta\} + ... + \frac{1}{2}.{}^nC_{\frac{n}{2}}]$$

Case II: n is odd.

When n is odd, then the number of terms are even and hence we have two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$.

$$T_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} x^{n-\frac{n-1}{2}}.\frac{1}{x_{\frac{n-1}{2}}} = {}^nC_{\frac{n-1}{2}} x.$$

$$T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} x^{n-\frac{n+1}{2}}.\frac{1}{x_{\frac{n+1}{2}}} = {}^nC_{\frac{n+1}{2}} \frac{1}{x}.$$

$$\therefore\ (2cos\theta)^n = {}^nC_0\left(x^n+\frac{1}{x^n}\right)+{}^nC_1\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+{}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right) + ... + {}^nC_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right)$$

$$\therefore\ 2^n cos^n\theta = {}^nC_0.(2cosn\theta)+{}^nC_1.\{2cos(n-2)\theta\}+{}^nC_2.\{2cos(n-4)\theta\} + ... + {}^nC_{\frac{n-1}{2}}2cos\theta$$

$$\therefore\ cos^n\theta = \frac{1}{2^{n-1}}[{}^nC_0.\{cosn\theta\} + {}^nC_1.\{cos(n-2)\theta\} + {}^nC_2.\{cos(n-4)\theta\} + ... + {}^nC_{\frac{n-1}{2}}cos\theta]$$

Note: Similarly we can find the expression for $sin^n\theta$ in terms of cosines or sines of multiples of θ according as n is even or odd by using $(2isin\theta)^n = \left(x-\frac{1}{x}\right)^n$.

Example 1.5.2. *Prove that*

$$cos^7\theta = \frac{1}{2^6}\left[cos7\theta + 7cos5\theta + 21cos3\theta + 35cos\theta\right]$$

.

Sol: Let $x = cos\theta + isin\theta$,

$$\therefore x^n = cosn\theta + isinn\theta$$

and $\frac{1}{x} = cos\theta - isin\theta$,

$$\frac{1}{x^n} = cosn\theta - isinn\theta$$

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$$\therefore x+\frac{1}{x}=2cos\theta, \quad x^n+\frac{1}{x^n}=2cosn\theta$$

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	1 7 21 35 35 21 7 1

$$\begin{aligned}\therefore (2cos\theta)^7 &= x^7+7x^5+21x^3+35x+\frac{35}{x}+\frac{21}{x^3}+\frac{7}{x^5}+\frac{1}{x^7}\\ &= \left(x^7+\frac{1}{x^7}\right)+7\left(x^5+\frac{1}{x^5}\right)+21\left(x^3+\frac{1}{x^3}\right)+35\left(x+\frac{1}{x}\right)\end{aligned}$$

$$\therefore 2^7.cos^7\theta = 2cos7\theta+7.2cos5\theta+21.2cos3\theta+35.2cos\theta$$

$$\implies cos^7\theta=\frac{1}{2^6}\left[cos7\theta+7cos5\theta+21cos3\theta+35cos\theta\right].$$

Example 1.5.3. *Prove that*

$$16sin^5\theta = sin5\theta - 5sin3\theta + 10sin\theta.$$

Sol: Let $x=cos\theta+isin\theta$, $\quad \therefore x^m = cosm\theta + isinm\theta$
and $\frac{1}{x}=cos\theta-isin\theta$, $\quad \therefore \frac{1}{x^m}=cosm\theta-isinm\theta$
$\therefore 2isin\theta = x-\frac{1}{x}, \quad 2isinm\theta = x^m-\frac{1}{x^m}$

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$$\therefore (2isin\theta)^5 = \left(x - \frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5} \quad \text{[By Pascal's Rule]}$$
$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$
$$\therefore 32isin^5\theta = 2isin5\theta - 5.2isin3\theta + 10.2isin\theta$$
$$or 16sin^5\theta = sin5\theta - 5sin3\theta + 10sin\theta$$

Example 1.5.4. *Prove that*

$$cos^6\,\theta sin^4\,\theta = 2^{-9}[cos\ 10\theta + 2cos\ 8\theta - 3cos\ 6\theta - 8cos\ 4\theta + 2cos\ 2\theta + 6]$$

Sol: Let $x = cos\theta + isin\theta$, $\quad \therefore \frac{1}{x} = cos\theta - isin\theta$
$\Longrightarrow\ 2cos\theta = x + \frac{1}{x},\ \ 2isin\theta = x - \frac{1}{x}$

$$\therefore \quad (2cos\theta)^6(2isin\theta)^4 = \left(x + \frac{1}{x}\right)^6\left(x - \frac{1}{x}\right)^4.$$

Now to obtain the coefficients of the various powers of x in the product, we first write the coefficients in the expansion of $\left(x + \frac{1}{x}\right)^6$, from Pascal's table, and then multiply by $x - \frac{1}{x}$ four times in succession.

1 6 15 20 15 6 1
1 -1

* * *

1 6 15 20 15 6 1
-1 -6 -15 -20 -15 -6 -1

* * *

I 1 5 9 5 -5 -9 -5 -1
1 -1

* * *

1 5 9 5 -5 -9 -5 -1
-1 -5 -9 -5 5 9 5 1

* * *

II 1 4 4 -4 -10 -4 4 4 1
1 -1

* * *

1 4 4 -4 -10 -4 4 4 1
-1 -4 -4 4 10 4 -4 -4 -1

* * *

III 1 3 0 -8 -6 6 8 0 -3 -1
1 -1

* * *

1 3 0 -8 -6 6 8 0 -3 -1
-1 -3 0 8 6 -6 -8 0 3 1

* * *

IV 1 2 -3 -8 2 12 2 -8 -3 2 1

These are the required coefficients.

Hence

$$\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^4$$
$$= x^{10}+2x^8-3x^6-8x^4+2x^2+12+\frac{2}{x^2}-\frac{8}{x^4}-\frac{3}{x^6}+\frac{2}{x^8}+\frac{1}{x^{10}}$$
$$= \left(x^{10}+\frac{1}{x^{10}}\right)+2\left(x^8+\frac{1}{x^8}\right)-3\left(x^6+\frac{1}{x^6}\right)-8\left(x^4+\frac{1}{x^4}\right)+2\left(x^2+\frac{1}{x^2}\right)+12$$

$$\therefore (2cos\theta)^6(2isin\theta)^4 = 2cos10\theta+2(2cos8\theta)-3(2cos6\theta)-8(2cos4\theta) + 2(2cos2\theta)+12$$

$$\therefore 2^{10}cos^6\theta sin^4\theta = 2[cos10\theta+2cos8\theta-3cos6\theta-8cos4\theta+2cos2\theta+6]$$
$$\therefore cos^6\theta sin^4\theta = 2^{-9}[cos10\theta+2cos8\theta-3cos6\theta-8cos4\theta+2cos2\theta+6]$$

Expanding $cosn\theta$ and $sinn\theta$ in terms of the t-ratios of θ, n being a positive integer.

By De-Moivre's Theorem,

$$(cos\theta+isin\theta)^n = cosn\theta+isinn\theta$$

or $cosn\theta+isinn\theta = (cos\theta+isin\theta)^n$

$$= {}^nC_0cos^n\theta+{}^nC_1cos^{n-1}\theta.(isin\theta)+{}^nC_2cos^{n-2}\theta(isin\theta)^2 + {}^nC_3cos^{n-3}\theta(isin\theta)^3+...+{}^nC_{n-1}cos\theta(isin\theta)^{n-1}+{}^nC_n(isin\theta)^n$$
$$= ({}^nC_0cos^n\theta-{}^nC_2cos^{n-2}\theta sin^2\theta+{}^nC_4cos^{n-4}\theta sin^4\theta-...) + i({}^nC_1cos^{n-1}\theta sin\theta-{}^nC_3cos^{n-3}\theta sin^3\theta+...)$$

Equating real and imaginary parts,

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$cosn\theta = {}^nC_0cos^n\theta - {}^nC_2cos^{n-2}\theta sin^2\theta + {}^nC_4cos^{n-4}\theta sin^4\theta - ...$
$sinn\theta = {}^nC_1cos^{n-1}\theta sin\theta - {}^nC_3cos^{n-3}\theta sin^3\theta + ...$

Cor:

$$\begin{aligned}\therefore tann\theta &= \frac{sinn\theta}{cosn\theta}\\ &= \frac{{}^nC_1cos^{n-1}\theta sin\theta - {}^nC_3cos^{n-3}\theta sin^3\theta + ...}{{}^nC_0cos^n\theta - {}^nC_2cos^{n-2}\theta sin^2\theta + {}^nC_4cos^{n-4}\theta sin^4\theta - ...}\\ &= \frac{{}^nC_1tan\theta - {}^nC_3tan^3\theta + {}^nC_5tan^5\theta - ...}{{}^nC_0 - {}^nC_2tan^2\theta + {}^nC_4tan^4\theta - ...}\end{aligned}$$

Example 1.5.5. *Expand $cos7\theta$ and $sin7\theta$ in powers of $cos\theta$ and $sin\theta$. hence, obtain an expression for $tan\theta$ in powers of $tan\theta$.*

Sol: By De-Moivre's Theorem

$$\begin{aligned}&cos7\theta + isin7\theta = (cos\theta + isin\theta)^7\\ &= {}^nC_70cos^7\theta + {}^nC_71cos^6\theta(isin\theta) + {}^nC_72cos^5\theta(isin\theta)^2 + {}^nC_73cos^4\theta(isin\theta)^3\\ &\quad + {}^nC_74cos^3\theta(isin\theta)^4 + {}^nC_75cos^2\theta(isin\theta)^5 + {}^nC_76cos\theta(isin\theta)^6\\ &\quad + {}^nC_77(isin\theta)^7\\ &= (cos^7\theta - 21cos^5\theta sin^2\theta + 35cos^3\theta sin^4\theta - 7cos\theta sin^6\theta) + i(7cos^6\theta sin\theta\\ &\quad - 35cos^4\theta sin^3\theta + 21cos^2\theta sin^5\theta - sin^7\theta)\end{aligned}$$

Equating real and imaginary parts,
$cos7\theta = cos^7\theta - 21cos^5\theta sin^2\theta + 35cos^3\theta sin^4\theta - 7cos\theta sin^6\theta$
$sin7\theta = 7cos^6\theta sin\theta - 35cos^4\theta sin^3\theta + 21cos^2\theta sin^5\theta - sin^7\theta$

$$
\begin{aligned}
\therefore tan7\theta &= \frac{sin7\theta}{cos7\theta} \\
&= \frac{7cos^6\theta sin\theta - 35cos^4\theta sin^3\theta + 21cos^2\theta sin^5\theta - sin^7\theta}{cos^7\theta - 21cos^5\theta sin^2\theta + 35cos^3\theta sin^4\theta - 7cos\theta sin^6\theta} \\
&= \frac{7tan\theta - 35tan^3\theta + 21tan^5\theta - tan^7\theta}{1 - 21tan^2\theta + 35tan^4\theta - 7tan^6\theta}
\end{aligned}
$$