Double points

Objective

To study double points.

Modules

Module I- Singular Points.

Module II- Double points.

Module III- Tangents at the origin.

Module I- Singular Points

Let $f(x, y) = 0$ be the given equation to the curve.

Differentiating with respect to x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

So the slope of the tangent to $f(x, y) = 0$ at (x, y) is

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

If $P(x, y)$ is any point on this curve and at least one of the partial derivatives $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ does not vanish, then at this point either $\frac{dy}{dx}$ or $\frac{dx}{dy}$ can be completely determined. That is the curve $f(x, y) = 0$ has a definite tangent at this point $P(x, y)$. In such a case, the point $P(x, y)$ is called an ordinary point.

If at some point $P(x, y)$, the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish, then $\frac{dy}{dx}$ is an indeterminate. In such a case, the point $P(x, y)$ is called a singular point.

Definition.

A point $P(x, y)$ on the curve $f(x, y) = 0$ is said to be a singular point if both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at this point.

Example 3. Consider the curve $(x - 2)^2 = y(y - 1)^2$.

Let $f(x, y) = (x - 2)^2 - y(y - 1)^2 = 0$.

Differentiating partially, w. r. to x and y, we get

$$\frac{\partial f}{\partial x} = 2(x - 2)$$

$$\frac{\partial f}{\partial y} = -[y2(y - 1) + (y - 1)^2]$$

.

Hence the possible singular points are

(2, 1), (2, 1/3).

As the point (2, 1/3) does not lie on the curve, the only singular point is (2, 1).

Module II- Double points

Definition.

A point on a curve through which more than one branch of the curve pass is called a multiple point.

If two branches of a curve pass through a point then the point is said to be **double point**.

The point is called a triple point, if three branches of a curve pass through the point.

In general, a multiple point is said to be of the r^{th} order if r branches of a curve pass through that point. In this case the curve has r tangents at this point.

Double points

There are two branches of a curve passing through a double point. Hence there are two tangents at this point. But these tangents may be real or imaginary.

Classification of double points

Double points are classified into three types depending on the nature of tangents i.e., whether the tangents are real or imaginary.

(i) Node: When the two tangents at a double point are real and distinct, the point is said to be a node.

(ii) Cusp: If the two tangents at a double point are real and coincident, then the point is said to be a cusp.

(iii) Conjugate or isolated point: If there no other real points of the curve in a neighbourhood of this point and the two tangents at a double point are imaginary, then the point is called a conjugate point.

Module III- Tangents at the origin

Let the equation of the curve through the origin be written in the form

$$(a_1x + a_2y) + (b_1x^2 + b_2xy + b_3y^2) + \cdots + (p_1x^n + p_2x^{n-1}y + \cdots + p_ny^n) \quad (1)$$

Note that the constant is absent in (1) as it passes through the origin.

Let $P(x, y)$ be any point on the curve (1).

The slope of the chord OP is $\frac{y}{x}$. As P tends to O along the curve, the chord OP tends to the tangent at the origin. That is, the slope of the tangent at O is

$$\lim_{x\to 0}\frac{y}{x}=m \text{(say)}, \qquad (2)$$

Assuming that the tangent is not the *y*- axis.

Hence the equation of the tangent to (1) at the origin is

$$\bar{y} = m\bar{x}$$

where *m* is given by (2).

Dividing (1) by *x* we obtain

$$\left(a_1 + a_2\frac{y}{x}\right) + \left(b_1x + b_2y + b_3y\frac{y}{x}\right) + \cdots = 0.$$
(3)

Taking limit as $x \to 0$ and using (2), we get from (3),

$$a_1 + a_2m = 0$$

i.e., $a_1\bar{x} + a_2\bar{y} = 0$ because $\bar{y} = m\bar{x}$.

Thus the equation of the tangent at the origin may be taken as

$$a_1x + a_2y = 0 \qquad (4)$$

which is same as the lowest degree terms in (1) equated to zero.

Now, if $a_2 = 0$, then by (4), $a_1 = 0$.

Therefore, if $a_1 = 0$ and $a_2 = 0$, then the equation (1) becomes

$$(b_1x^2 + b_2xy + b_3y^2) + (c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3) + \cdots = 0$$
(5)

Dividing by x_2, taking limit as tends to 0 and using (2), we obtain

$$b_1 + b_2m + b_3m^2 = 0.$$

Since $m = \frac{\bar{y}}{\bar{x}}$, we get

$$b_1\bar{x}^2 + b_2\overline{xy} + b_3\bar{y}^2 = 0$$

Or $$b_1x^2 + b_2xy + b_3y^2 = 0 \quad (6)$$

which represents a pair of tangents at the origin.

Similarly, we can show that if

$$a_1 = a_2 = b_1 = b_2 = b_3 = 0,$$

then $$c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 = 0$$

gives the combined equation of the three tangents at the origin.

Generalising this we deduce:

The tangents at the origin are given by equating to zero the lowest degree terms in the rational algebraic equation of the given curve.

Remark 1: If the *y*- axis is a tangent at the origin we can easily see that, by interchanging the axes of *x* and *y*, the rule is still true.

Remark 2: The above process can also be used to find the equation to the tangent (or tangents) at any point (h, k) on the curve. This can be done by shifting the origin to this point (h, k) and then applying the above rule.

The necessary condition for the existence of a double point at any point $P(x, y)$ of the curve $f(x, y) = 0$ is that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Procedure for finding the nature of the double points

Let (h, k) be a double point.

Shift the origin to (h, k) by using the transformation equations

$$x = \overline{x} + h, y = \overline{y} + k.$$

Find the tangents at the origin by equating to zero the lowest degree terms in the transformed equation. If the tangents are real and distinct, then the new origin is a node.

However, in the case of cusp we must show that the branches of the curves near the origin are real.

If the tangents are coincident, it is a cusp.

If the tangents are imaginary, it is a conjugate point.

Example 4. Consider the curve $f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

Here

$$\frac{\partial f}{\partial x} = 3x^2 - 14x + 15$$

and $\frac{\partial f}{\partial y} = -2y + 4$

The coordinates of double points are obtained by solving the equations

$$3x^2 - 14x + 15 = 0 \text{ and } -2y + 4 = 0$$

$y = 2$ and $x = 3$, $x = 5/3$.

The possible double points are (3, 2) and (5/3, 2). But (5/3, 2) does not satisfy the given equation.

Hence the only double point is (3, 2).

Shift the origin to the point (3, 2) by putting $x = \overline{x} + 3$ and $y = \overline{y} + 2$ in the given equation, we get

$$\overline{x}^3 - \overline{y}^2 + 2\overline{x}^2 = 0.$$

Equations of tangents at the new origin (3, 2) (i.e., the double point) are given by

$$-\overline{y}^2 + 2\overline{x}^2 = 0$$

implies $\overline{y} = \pm\sqrt{2}\,\overline{x}$ or $y - 2 = \pm\sqrt{2}(x - 3)$.

Since the two tangents are real and distinct the double point (3, 2) is a node.

Summary

A curve $y = f(x)$ (i) is concave upwards in $[a,b]$ if $f''(x) > 0, \forall\, x \in [a,b]$, (ii) is concave downwards in $[a,b]$ if $f''(x) < 0, \forall\, x \in [a,b]$ and (iii) has a point of inflexion at $[c, f(c)]$ if $f''(c) = 0$ and $f''(x)$ changes sign as x passes through c.

A point $P(x, y)$ on the curve $f(x, y) = 0$ is said to be a singular point if both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at this point.

A point on a curve through which more than one branch of the curve pass is called a multiple point.

If two branches of a curve pass through a point then the point is said to be double point.

A multiple point is said to be of the r^{th} order if r branches of a curve pass through that point. In this case the curve has r tangents at this point.

Double points are classified into three types namely, node, cusp and isolated point depending on the nature of tangents.

The tangents at the origin are given by equating to zero the lowest degree terms in the rational algebraic equation of the given curve.

The necessary condition for the existence of a double point at any point $P(x, y)$ of the curve $f(x, y) = 0$ is that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Assignment Questions

1. Show that the origin is a node; a cusp or a conjugate point on the curve $y^2 = ax^2 + ax^3$.

2. Find the position and nature of the double points on the curve $y^2 = x^2(x - 1)$.

Quiz Questions

1. The curve $(a^2 + x^2)y = a^2x$ has……….. points of inflexions

   a. Two

   b. Three

   c. No

2. The tangents at the origin in a rational algebraic equation are given by equating to zero

   a. the constant terms.

   b. the highest degree terms.

   c. the lowest degree terms.

3. If the tangents at a double point are coincident then

   a. it is a cusp.

   b. it is a node.

   c. it is an isolated point.

4. For the curve $f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$, the only double point is

   a. (3, 2)

   b. (5/3, 2)

   c. (2, 3)

Answers

1. b 2. c 3. a 4. b